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Supersymmetry and Bogomol'nyi equations in the Abelian Higgs Model

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Abstract

The $N = 2$ supersymmetric extension of the $2+1$ dimensional Abelian Higgs model is discussed. By analysing the resulting supercharge algebra, the connection between supersymmetry and Bogomol'nyi equations is clarified. Analogous results are presented when the model is considered in 2-dimensional (Euclidean) space.

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1 Introduction

Some twenty years ago Belavin, Polyakov, Schwartz and Tyupkin [1] found regular solutions to the four-dimensional Euclidean Yang-Mills equations of motion (the well-honoured instantons) by showing that the Yang-Mills action is bounded from below. When the bound, which is of topological nature, is saturated, solutions can be constructed by studying first order differential equations instead of the more involved second order Euler-Lagrange equations.

Almost inmediately, other systems exhibiting first order differential equations with solutions satisfying second order equations of motion were investigated [2, 3]. They correspond to gauge theories with spontaneous symmetry breaking and the solutions were (static) solitons (vortices, monopoles) saturating a bound of topological character, this time for the energy. More recently, Bogomol'nyi equations for the Chern-Simons-Higgs system were also found [4]-[5]. Appart from the gauge coupling constant, symmetry breaking implies the introduction of a second coupling constant for the Higgs potential. Strikingly enough, the topological bound and the resulting first-order differential equations (known today as Bogomol'nyi equations) require certain conditions on these coupling constants which reappear in other apparently distinct fields.

Indeed, already in ref.[2], de Vega and one of the authors of the present paper stressed that the required relation between coupling constants in the Abelian Higgs model (which corresponds to the limit between type-I and type-II superconductivity in the Ginsburg Landau model) was precisely the same needed for its supersymmetric extension [6]-[7].

Although the connection between vortex solutions and supersymmetry in the Abelian Higgs model was afterwards thoroughly studied [8], the reasons behind the overlap of these two apparently divorced matters (supersymmetry and topological bounds) were not investigated till very recently [9]-[10]. Much was understood in these last works on the connection by analysing several models but the case in which the coincidence was first stressed, i.e. the Abelian Higgs model, has not been detaily discussed yet.

It is the purpose of this work to study this issue, eventually finding the reasons behind the coincident conditions imposed on the Abelian Higgs model by supersymmetry and the existence of Bogomol'nyi equations . Originally, these first order equations were studied for static, axially symmetric configurations. They could be interpreted as solitons (Nielsen-Olesen vortices [11]). In the approach of Hlousek-Spector [9]-[10], the identification of the topological charge for the vortex configuration (its magnetic flux) with the $N = 2$ supersymmetry central charge plays a central rôle. Since in the Abelian Higgs model the topological charge is defined in 2-dimensional space, we shall first work, taking advantage of axial symmetry, in (2+1) space-time dimensions so that charges are given by two-dimensional integrals. Now, as it is well known, vortices in $(2 + 1)$ dimensions can be interpreted as instantons in 2-dimensional Euclidean space. Although the general analysis of refs.[9]-[10] does not apply to 2-dimensional models, we shall also study the Abelian Higgs model as a bidimensional model showing that again in this case the arguments connecting supersymmetry and Bogomol'nyi equations can be applied.

Our results can be summarized as follows: starting from the Abelian Higgs model in $2+1$ dimensions for which a topological charge can be found and an $N = 1$ supersymmetric extension can be constructed, an $N = 2$ supersymmetric extension can be obtained provided a relation between coupling constants holds. The same relation is required for the existence of Bogomol'nyi equations. Similar results hold when the Higgs model is defined in 2 dimensional Euclidean space. The connection between $N = 2$ supersymmetry and Bogomol'nyi equations is clarified by the explicit construction of the supercharge algebra which leads to a bound for the energy in terms of the central charge (which coincides with the topological charge). The bound is saturated by solutions of Bogomol'nyi equations. In conclusion, in models with gauge symmetry breaking, $N = 2$ supersymmetry forces a relation between coupling constants and at the same time, through its supercharge algebra, imposes Bogomol'nyi equations on field configurations.

2 The Model

The Abelian Higgs model dynamics in (2+1)-Minkowski space is defined by the action:

$$\mathcal{S}_{\mathcal{H}} = \int d^3x \{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - \lambda (|\phi|^2 - \phi_0^2)^2 \} \tag{1}$$

Here ϕ is a complex scalar,

$$\phi = \phi^1 + i\phi^2, \tag{2}$$

the covariant derivative is defined as

$$D_\mu = \partial_\mu + ieA_\mu \tag{3}$$

and $g^{\mu\nu} = (+--)$. An $N=1$ supersymmetric extension of this model is given by the action:

$$\begin{aligned}
\mathcal{S}_{N=1} &= \int d^3x\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu M)(\partial^\mu M) + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - 2\lambda M^2|\phi|^2 \\
&- \lambda(|\phi|^2 - {\phi_0}^2)^2 + \frac{i}{2}\overline{\rho}\ \partial\!\!\!/\rho + \frac{i}{2}\overline{\chi}\ \partial\!\!\!/\chi + \frac{i}{2}\overline{\psi}\ D\!\!\!\!/\psi - (2\lambda)^{1/2}M\overline{\psi}\psi \\
&+ \frac{ie}{2}(\overline{\psi}\rho\phi - \overline{\rho}\psi\phi^*) - (2\lambda)^{1/2}(\overline{\psi}\chi\phi + \overline{\chi}\psi\phi^*)\}
\end{aligned} \tag{4}$$

Our conventions for γ-matrices, $(\gamma^\mu)_\alpha{}^\beta$ are,

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{5}$$

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma_\lambda$$

In (4) ρ and χ are Majorana fermions, ψ is a Dirac fermion and M is a real scalar field. In fact, action (4) can be constructed by considering a complex scalar superfield $\Phi = (\phi, \psi, F)$, a real spinor superfield $\Gamma^\alpha = (A_\mu, \rho, \xi, \varphi)$ and a real superfield $S = (M, \chi, D)$. Here F and D are auxiliary fields which can be eliminated using their equations of motion. Concerning ξ and φ they are absent in the Wess-Zumino gauge which we adopt from here on. It is easily seen that action (4) is invariant under the following supersymmetry transformations:

$$\begin{aligned}
\delta\rho &= -\frac{i}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda\eta \quad , \quad \delta A_\mu = -i\overline{\eta}\gamma_\mu\lambda \quad , \quad \delta\psi = -i\gamma^\mu\eta D_\mu\phi - (8\lambda)^{1/2}M\phi\eta \\
\delta\chi &= -(2\lambda)^{1/2}(|\phi|^2 - {\phi_0}^2)\eta - i\gamma^\mu\eta\partial_\mu M \quad , \quad \delta M = \overline{\eta}\chi \quad , \quad \delta\phi = \overline{\eta}\psi
\end{aligned} \tag{6}$$

where η is an infinitesimal Majorana spinor. Note that supersymmetry is achieved without imposing a relation between e and λ as required in refs.[6, 7].

The corresponding spinor supercurrent $\mathcal{J}^\mu_{N=1}$ is given by:

$$\begin{aligned}
\mathcal{J}^\mu_{N=1} &= \frac{1}{4}\overline{\rho}\gamma^\mu\epsilon^{\nu\lambda\sigma}F_{\nu\lambda}\gamma_\sigma - i\left(\frac{\lambda}{2}\right)^{1/2}\overline{\chi}\gamma^\mu(|\phi|^2 - {\phi_0}^2) + \frac{1}{2}\overline{\chi}\gamma^\mu\ \partial\!\!\!/M \\
&+ \frac{1}{2}\overline{\psi}\gamma^\mu\ D\!\!\!\!/\phi - i\overline{\psi}\gamma^\mu(2\lambda)^{1/2}M\phi
\end{aligned} \tag{7}$$

We now want to impose the requirement that the theory be invariant under an $N = 2$ extended supersymmetry. This can be achieved by considering transformations (6) with complex parameter η_c (now an infinitesimal Dirac spinor). Being ρ and χ real fermions, we combine them into a Dirac fermion Σ,

$$\Sigma \equiv \chi - i\rho \tag{8}$$

so that, transformations (6) become

$$\begin{aligned}
\hat{\delta}\Sigma &= -\left(\frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda + (2\lambda)^{1/2}(|\phi|^2 - {\phi_0}^2) + i\,\partial\!\!\!/ M\right)\eta_c \quad , \quad \hat{\delta}A_\mu = -i\overline{\eta}_c\gamma_\mu\lambda \\
\hat{\delta}\psi &= -i\gamma^\mu D_\mu\phi\eta_c - (8\lambda)^{1/2}M\phi\eta_c \quad , \quad \hat{\delta}M = \overline{\eta}_c\chi \quad , \quad \hat{\delta}\phi = \overline{\eta}_c\psi
\end{aligned} \tag{9}$$

Using (8), action (4) can be rewritten in the form

$$\begin{aligned}
\mathcal{S}_{N=1} &= \int d^3x\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu M)(\partial^\mu M) + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - 2\lambda M^2|\phi|^2 \\
&- \lambda(|\phi|^2 - {\phi_0}^2)^2 + \frac{i}{2}\overline{\Sigma}\,\partial\!\!\!/\Sigma + \frac{i}{2}\overline{\psi}\,D\!\!\!\!/\psi - (2\lambda)^{1/2}M\overline{\psi}\psi \\
&- \frac{e + (8\lambda)^{1/2}}{4}(\overline{\psi}\Sigma\phi + h.c.) + \frac{e - (8\lambda)^{1/2}}{4}(\overline{\psi}\Sigma^c\phi + h.c.)
\end{aligned} \tag{10}$$

Here Σ^c is the charge conjugate (the complex conjugate) of Σ.

Now, transformations (9) with complex parameter $\eta_c = \eta e^{-i\alpha}$ are equivalent to transformations (6) with real parameter η followed by a phase transformation for fermions Σ ($\Sigma \to e^{i\alpha}\Sigma$) and ψ ($\psi \to e^{i\alpha}\psi$). Then, $N = 2$ supersymmetry requires invariance under this fermion rotation. One can easily see that fermion phase rotation invariance is achieved if and only if:

$$\lambda = \frac{e^2}{8} \tag{11}$$

Indeed, under a phase rotation, the factor $\overline{\psi}\Sigma^c\phi$ ($[\overline{\psi}\Sigma^c\phi]^\dagger$) appearing in the last term in (10) picks a $e^{-2i\alpha}$ ($e^{+2i\alpha}$) phase, so that condition (11) is needed in order to make the last term to vanish. In this way phase rotation invariance is ensured or, what is the same, $N = 2$ supersymmetry invariance is achieved. This corresponds to a well-known result holding in general when, starting from an $N = 1$ supersymmetric gauge model, one attemps to impose a second supersymmetry [12]: conditions on coupling constants have to be imposed so as to accommodate different $N=1$ multiplets into an $N=2$ multiplet. Eq.(11) is an example of such a condition. In fact, condition (11)was obtained by di Vecchia and Ferrara [8] when looking for the $N=2$ supersymmetry extension of the 2-dimensional Abelian Higgs model.

In summary we have arrived to the following $N=2$ supersymmetric action associated with the Abelian Higgs model:

$$\begin{aligned}
\mathcal{S}_{N=2} &= \int d^3x\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(\partial_\mu M)(\partial^\mu M)+\frac{1}{2}(D_\mu\phi)^*(D^\mu\phi)-\frac{e^2}{4}M^2|\phi|^2 \\
&- \frac{e^2}{8}(|\phi|^2-{\phi_0}^2)^2+\frac{i}{2}\overline{\Sigma}\ \partial\!\!\!/\Sigma+\frac{i}{2}\overline{\psi}\ D\!\!\!\!/\psi-\frac{e}{2}M\overline{\psi}\psi \\
&- \frac{e}{2}(\overline{\psi}\Sigma\phi+h.c.)
\end{aligned} \tag{12}$$

In the next section, the reasons why condition (11) ensuring $N=2$ supersymmetry is also needed in order to attain the Bogomol'nyi bound will be clear at the light of Hlousek-Spector approach [9]-[10].

3 Supercharge and Bogomol'nyi equations

$N=2$ supercharges generating the supersymmetry transformations (9) can be found from the conserved quantity

$$\mathcal{Q}=\frac{\sqrt{2}}{e\phi_0}\int d^2x\mathcal{J}^0_{N=2} \tag{13}$$

where $\mathcal{J}^\mu_{N=2}$ is the Noether current associated with transformations (9). Writing

$$\mathcal{Q}=\overline{\eta}_cQ+\overline{Q}\eta_c \tag{14}$$

we find

$$\begin{aligned}
Q &= \frac{\sqrt{2}}{e\phi_0}\int d^2x[\left(-\frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda+i\ \partial\!\!\!/M-\frac{e}{2}(|\phi|^2-{\phi_0}^2)\right)\gamma^0\Sigma \\
&+ \left(i(D\!\!\!\!/\phi)^*-\frac{e}{2}M\phi^*\right)\gamma^0\psi]
\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
\overline{Q} &= \frac{\sqrt{2}}{e\phi_0}\int d^2x[\overline{\Sigma}\gamma^0\left(-\frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda - i\,\partial\!\!\!/ M - \frac{e}{2}(|\phi|^2-{\phi_0}^2)\right) \\
&+ \overline{\psi}\gamma^0\left(-i\,D\!\!\!\!/\phi - \frac{e}{2}M\phi\right)]
\end{aligned} \tag{16}
$$

We are interested in connecting the supercharge algebra and Bogomol'nyi equations for vortices of the original Abelian Higgs model. We then restrict the model to its bosonic sector by putting $M = 0$ and, after computing the supercharge algebra, all fermions to zero so as to end with the original Abelian Higgs model. Moreover, since Bogomol'nyi equations correspond to static configurations with $A_0 = 0$, we impose these conditions on (15)-(16), finding for the anticommutation relation among spinor supercharges Q , $\overline{Q}$:

$$
\{Q_\alpha, \overline{Q}^\beta\} = 2(\gamma_0)_\alpha{}^\beta P^0 + \delta_\alpha{}^\beta T \tag{17}
$$

where

$$
P^0 = E = \frac{1}{2e^2{\phi_0}^2}\int d^2x\left[\frac{1}{2}F_{ij}^2 + |D_i\phi|^2 + \frac{e^2}{4}(|\phi|^2-{\phi_0}^2)^2\right] \tag{18}
$$

while the central charge is given by:

$$
T = -\frac{1}{e^2{\phi_0}^2}\int d^2x\left[\frac{e}{2}\epsilon^{ij}F_{ij}(|\phi|^2-{\phi_0}^2) + i\epsilon^{ij}(D_i\phi)(D_j\phi)^*\right] \tag{19}
$$

Here $i, j = 1, 2$.

We are now ready to find the connection between $N = 2$ supersymmetry and Bogomol'nyi equations. Let us start by noting that the central charge in (19) coincides with the topological charge of the Abelian Higgs model. Indeed, T can be rewritten in the form

$$
T = \int \partial_i \mathcal{V}^i d^2x \tag{20}
$$

where $\mathcal{V}^i$ is given by

$$
\mathcal{V}^i = \left(\frac{1}{e}A_j + \frac{i}{e^2{\phi_0}^2}\phi^* D_j\phi\right)\epsilon^{ij} \tag{21}
$$

so that, after Stokes theorem (and taking into account that $D_i\phi \to 0$ at infinity)

$$T = \frac{1}{e}\oint A_i dx^i = \frac{2\pi n}{e} \tag{22}$$

where $n, n \in Z$, is the integer characterizing the homotopy class to which A_i belongs.

This sort of identity between the $N = 2$ central charge and topological charge was first stressed by Olive and Witten [13] in their study of the $SO(3)$ Georgi-Glashow model in the Prasad-Sommerfield limit. It was also discussed for the self-dual Chern-Simons system by Lee, Lee and Weinberg [14]. More recently, Hlousek and Spector [9]-[10] have thoroughly analysed this conection by studying several models where the existence of an $N = 1$ supersymmetry and a topological current implies an $N = 2$ supersymmetry with its central charge coinciding with the topological charge. It is important to stress at this point that in order to achieve the $N = 2$ supersymmetry one is forced to impose the condition (11) exactly as it happens when trying to find a Bogomol'nyi bound for the Abelian Higgs model. This condition is unavoidable both for having $N = 2$ supersymmmetry and Bogomol'nyi equations. Also, in the study of self-dual Chern-Simons systems, for which a topological charge (related to the magnetic flux) and an $N = 1$ extension does exist, a condition on the symmetry breaking coupling constant must be imposed both to achieve $N = 2$ extended supersymmetry [14] and to obtain Bogomolnyi equations [4],[5].

At the light of the discussion above, it should be worthwhile to formulate the result in [9]-[10], when applied to models with symmetry breaking by stating that:

For gauge theories with spontaneous symmetry breaking and a topological charge with an $N = 1$ supersymmetric version, the N=2 supersymmetric extension, which requires certain conditions for coupling constants, has a central charge coinciding with the topological charge

It is now easy to find [9]-[10] the Bogomol'nyi bound from the supersymmetry algebra (17). Indeed, since the anticommutators in (17) are Hermitian, one has:

$$\{Q_\alpha, \overline{Q}^\beta\}\{Q^\alpha, \overline{Q}_\beta\} \geq 0 \tag{23}$$

or using (17),

$$E \geq |T| \tag{24}$$

Now, in view of (18) and (19), one finds Bogomol'nyi result:

$$E \geq \frac{2\pi}{e}|n| \tag{25}$$

In order to explicitly obtain Bogomol'nyi equations (saturating the energy bound) from the supersymmetry algebra, we define, following Hlousek and Spector [9]-[10]:

$$Q_I = \frac{Q_+ + iQ_-}{\sqrt{2}} \tag{26}$$

$$Q_{II} = \frac{\overline{Q}^+ + i\overline{Q}^-}{\sqrt{2}} \tag{27}$$

where we have defined $Q_\pm$ from

$$Q = \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix} \tag{28}$$

$$\overline{Q} = \begin{pmatrix} \overline{Q}^+ & \overline{Q}^- \end{pmatrix} \tag{29}$$

Now, suppose that a field configuration $|B>$ saturates the Bogomol'nyi bound derived from (23). Then, one necessarily has

$$(Q_I \pm Q_{II})\,|B>= 0 \tag{30}$$

or, using (26)-(29) and (15)-(16)

$$\begin{aligned} \epsilon^{ij}F_{ij} &= \pm e(|\phi|^2 - {\phi_0}^2) \\ i\epsilon_{ij}D^i\phi &= \pm(D_j\phi)^* \end{aligned} \tag{31}$$

These are of course the Bogomol'nyi equations for the Abelian Higgs model [2]-[3]. Due to (25), their solution also solves the static Euler-Lagrange equations of motion. Let us insist that the condition (11) necesary for this last fact, arises in the present approach from the requirement of $N = 2$ supersymmetry.

In conclusion, in models with gauge symmetry breaking, like the Abelian Higgs model, $N = 2$ supersymmetry forces a relation between coupling constants and at the same time, through its supercharge algebra, imposes Bogomol'nyi equations on field configurations.

4 The Model in d=2 dimensions

The Hlousek-Spector [9]-[10] approach was basically formulated in (2+1) and (3+1) dimensions so that our (2+1) discussion above enters in this family of models. Concerning (1+1) models, although no general results are presented in refs.[9, 10], connection between supersymmetry and Bogomol'nyi equations are well-known in this case [8]-[13],[9]. We would like to show in this section that results analogous to those obtained in section 3 can be found if one considers the Abelian Higgs model in $(1+1)$ Euclidean dimensions where vortex configurations become instantons.

Following di Vecchia and Ferrara [8], the $N = 1$ supersymmetry model to consider in $d = 2$ dimensions is (in the Wess-Zumino gauge)

$$\begin{aligned} S_{N=1} &= \int d^2x\{-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu P)^2 - \frac{1}{2}(\partial_\mu M)^2 - \frac{1}{2}(\mathcal{D}_\mu\vec{\Phi})^2 + \frac{1}{2}e^2P^2\vec{\Phi}^2 \\ &+ \frac{i}{2}\rho\ \partial\!\!\!/\rho + \frac{i}{2}\vec{\psi}\cdot\mathcal{D}\!\!\!\!/\vec{\psi} + \frac{i}{2}\chi\ \partial\!\!\!/\chi - \frac{ie}{2}\vec{\psi}\wedge\vec{\psi}P + ie\vec{\psi}\wedge\gamma_5\rho\vec{\Phi} \\ &- i\sqrt{2\lambda}M\vec{\psi}\cdot\gamma_5\vec{\psi} - i2\sqrt{2\lambda}\vec{\Phi}\cdot\vec{\psi}\gamma_5\chi - 4\lambda M^2\vec{\Phi}^2 - \lambda(\vec{\Phi}^2 - {\Phi_0}^2)^2\} \end{aligned} \tag{32}$$

where $\vec{\Phi}$ is the (real) Higgs field doublet which is in the adjoint representation of $SO(2)$, $\vec{\psi}$ is a doublet of Majorana fermions, ρ and χ are Majorana fermions and M (P) is a real scalar (pseudoscalar). The γ-matrices are taken as:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It is understood that $\vec{\psi}\cdot\vec{\Phi}$ means $\psi^a\Phi^a$ while $\vec{\psi}\wedge\vec{\Phi}$ is the (pseudo)scalar $\epsilon^{ab}\psi^a\Phi^b$.

It is interesting to note that action (33) can be obtained by dimensional reduction (including identification of A_0 in $d = 3$ with P in $d = 2$) of action (4). Hence, $N = 1$ supersymmetry transformations can simply be inferred from transformations (6):

$$\begin{aligned} \delta\Phi^a &= i\epsilon\psi^a \quad , \quad \delta\chi = (\partial\!\!\!/M - \sqrt{2\lambda}(\vec{\Phi}^2 - {\Phi_0}^2)\gamma_5)\epsilon \quad , \quad \delta P = -\frac{i}{2}\epsilon\gamma_5\rho \\ \delta A_\mu &= -\frac{i}{2}\epsilon\gamma_5\gamma_\mu\rho \quad , \quad \delta\psi^a = (\mathcal{D}\!\!\!\!/\Phi^a - 2\sqrt{2\lambda}\gamma_5 M\Phi^a + eP\epsilon^{ab}\Phi^b)\epsilon \\ \delta\rho &= (\gamma_5\ \partial\!\!\!/P - \frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu})\epsilon \quad , \quad \delta M = i\epsilon\chi \end{aligned} \tag{33}$$

It is evident that condition (11) must hold also in the present $d = 2$ model in order to have $N = 2$ extended supersymmetry, as was first pointed out by di Vecchia-Ferrara [8]. As discussed previously, it ensures that fermions χ and ρ can be accommodated into a unique $N = 2$ supermultiplet. If we call $\Sigma^1 = \rho$ and $\Sigma^2 = \gamma_5\chi$, and impose condition (11), the resulting $N = 2$ supersymmetry transformations for any field ϕ read

$$\hat{\delta}\phi = \delta_1\phi\eta^1 + \delta_2\phi\eta^2 \tag{34}$$

with $\eta_c = \eta_1 + i\eta_2$, the $N = 2$ transformation parameter. In component fields, eq.(34) is:

$$\begin{aligned}
\delta_1 M &= -i\gamma_5\Sigma^2 \quad , \quad \delta_2 M = -i\gamma_5\Sigma^1 \quad , \quad \delta_1\Phi^a = i\psi^a \quad , \quad \delta_2\Phi^a = -i\epsilon^{ab}\gamma_5\psi^b \\
\delta_1 A_\mu &= -i\gamma_5\gamma_\mu\Sigma^1 \quad , \quad \delta_2 A_\mu = -i\gamma_5\gamma_\mu\Sigma^2 \quad , \quad \delta_a P = -i\gamma_5\Sigma^a \\
\delta_1\psi^a &= \not{\mathcal{D}}\Phi^a + eP\gamma_5\Phi^a - eM\epsilon^{ab}\Phi^b \\
\delta_2\psi^a &= \epsilon^{ab}\gamma_5\not{\mathcal{D}}\Phi^b - eM\gamma_5\Phi^a + eP\epsilon^{ab}\Phi^b \\
\delta_a\Sigma^a &= -(-1)^a\gamma_5\not{\partial}P - \frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu} \\
\delta_a\Sigma^b &= \epsilon^{ab}\gamma_5\not{\partial}M + \frac{e}{2}(\vec{\Phi}^2 - {\Phi_0}^2)
\end{aligned} \tag{35}$$

At this point let us note that conditions (30) leading to Bogomol'nyi equations, can be seen to be equivalent to:

$$\begin{aligned}
(\delta_1 \pm \delta_2)\vec{\psi} = 0 \\
(\delta_1 \pm \delta_2)\vec{\Sigma} = 0
\end{aligned} \tag{36}$$

Now, it is easy to see that (36) implies:

$$P = \mp M \tag{37}$$

$$\epsilon_{\mu\nu}F_{\mu\nu} = \pm e(\vec{\Phi}^2 - {\Phi_0}^2) \tag{38}$$

$$\epsilon_{\mu\nu}\epsilon^{ab}\mathcal{D}_\nu\Phi^b = \pm\mathcal{D}_\mu\Phi^a \tag{39}$$

We can recognize eqs.(38)-(39) as the usual self-dual equations whose solutions saturate a bound of the action. It can be seen that the other fields entering in the $N = 2$ supersymmetric model also satisfy Bogomol'nyi equation. Indeed, following the same procedure, one can impose to bosonic fields:

$$(\delta_1 \pm \delta_2)\phi = 0 \tag{40}$$

so that one obtains:

$$\rho = \pm\gamma_5\chi \tag{41}$$
$$\psi^a = \pm\epsilon^{ab}\gamma_5\psi^b \tag{42}$$

which are the fermionic superpartner of Bogomol'nyi equations appearing above [8].

To conclude, we would like to point that the connection between supersymmetry and self-duality equations might be of interest in the study of gravity. In particular, a $d = 2$ model for gravity coupled to matter can be constructed from a topological model where self-duality equations play a central role [15]-[16]. In this context, as well as in the study of black hole solutions [17], supersymmetry should provide a natural framework to analyse classical and quantum properties. We hope to report on these issues in a forthcoming work.

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